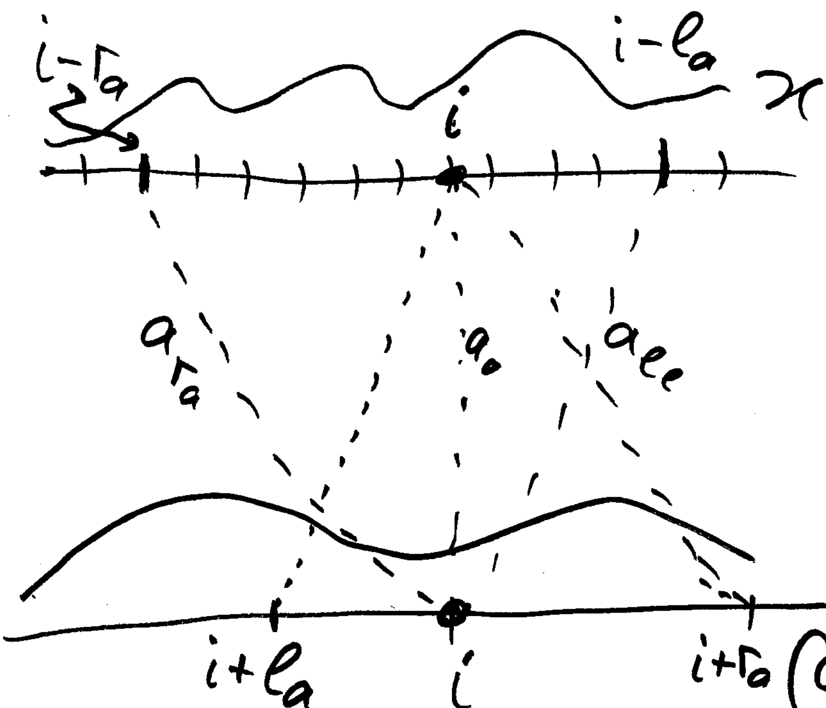
Signals (vectors) – functions on $\mathbb{Z}$

Distortion (A) convolution with a – convolution kernel – point spread function.

$$\Delta_a = [l_a, r_a] = \{l_a, l_a+1, \ldots r_a\}$$



i influences $i+\Delta_a$

$$Ax = a * x$$

$$(a*x)_i = \sum_{j\in\Delta_a} a_j x_{i-j}$$

i is influenced by $i-\Delta_a = [i-r_a, i-l_a]$

δ – function $= 1$ at 0, and 0 everywhere else

$$\delta_i = \begin{cases} 1 & i=0 \\ 0 & i\neq 0 \end{cases} \qquad \Delta_\delta = \{0\} = [0,0]$$

$$a*\delta = a = \delta * a.$$

δ – neutral element wrt convolution

If ν - random field and $\operatorname{cov}\nu = s$ - its covariance function

$$s_i = E\nu_k \nu_{k+i} \qquad \operatorname{Var} \nu_k = s_0 = E\nu_k^2$$

If components of ν are ind.

$s = \sigma^2 \delta$ white noise.

$$\mu = a * \nu \Rightarrow \operatorname{cov}\mu = a * s * a^*$$

If $s = \sigma^2 \delta$

$$\operatorname{cov}\mu = a * (\sigma^2 \delta) * a^* =$$

$$= \sigma^2 (\underbrace{a * \delta}_{=a} * a^*) = \sigma^2 \cdot a * a^*$$

Linear experiment

$$y = Ax + v \iff y = a * x + v$$

a - given

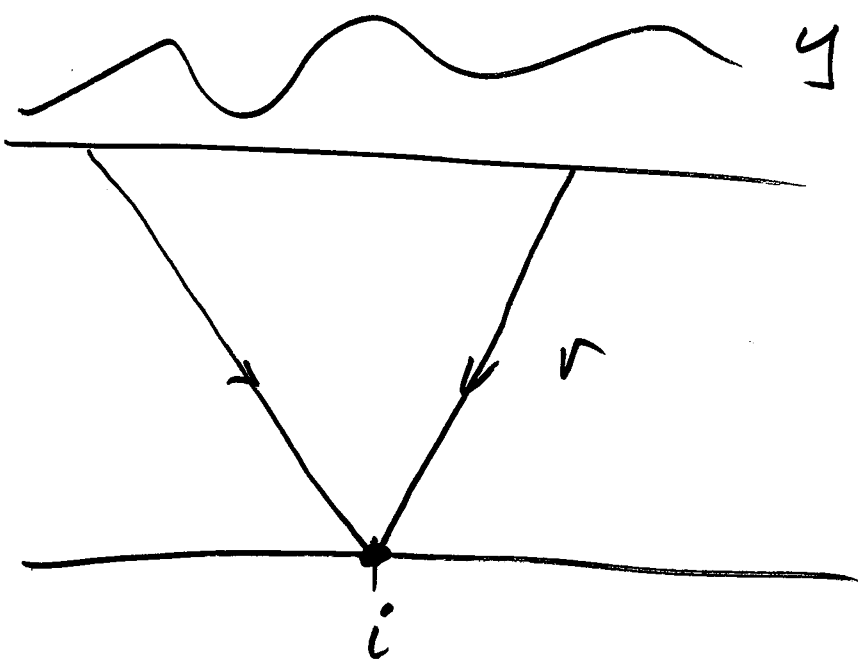
$Ev = 0 \quad \operatorname{cov} v = S$ - given.

Prior info about x.

$Ex = 0 \quad \operatorname{cov} x = f$ - given.

x and v - indep.

Want to estimate x. $\hat{x} = ?$



$\hat{x} = Ry$

represent R with PSF r

Choose Δ

Find best $\hat{x}$ r with $\Delta_r = \Delta$

$$\hat{x} - x = r * y - x = r * (a * x + v) - x$$
$$= (r * a - \delta) * x + r * v$$

$$\hat{x} - x = Ry - x = R(Ay + v) - x$$
$$= (RA - I)x + Rv$$

$$M(r) = E(\hat{x}_i - x_i)^2 = E\left[(r*a-\delta)*x + r*v\right]_i^2$$

$$= E\left((r*a-\delta)*x\right)_i^2 + E\left(r*v\right)_i^2$$

$$= \left[(r*a-\delta)*f*(r*a-\delta)^*\right]_0 + \left[r*s*r^*\right]_0$$

$$= \Big[r*a*f*a^**r^* - f*a^**r^* - r*a*f$$
$$+ f + r*s*r^*\Big]_0$$

$$= \Big[r*\underbrace{(a*f*a^*+s)}_{p}*r^* - \overbrace{f*a^*}^{q}r^*$$
$$- r*\underbrace{a*f}_{q^*} + f\Big]_0$$

$$\begin{cases} p = a*f*a^*+s \\ q = f*a^* \end{cases}$$

$$= (r*p*r^*)_0 - (q*r^*)_0 - (r*q^*)_0 + f_0 \Big)_0$$

want r! with $\Delta_r = \Delta$:

$$M(r) \sim \min_r$$

$$(q * r^*)_0 = \sum_{k\in\Delta} q_k \underbrace{r^*_{0-k}}_{=r_k} = \sum_{k\in\Delta} q_k r_k$$

$$= \langle q_\Delta, r\rangle$$

where q_Δ - restriction of q to Δ

$$(r * q^*)_0 = \sum_{k\in\Delta} r_k q^*_{0-k} = \langle r, q_\Delta\rangle$$

$$(r * p * r^*)_0 = \sum_k (r * p)_k \underbrace{r^*_{0-k}}_{=r_k} =$$

$$= \sum_{k\in\Delta}\sum_{m\in\Delta} r_m \underbrace{p_{k-m}}_{P_{km}} r_k = \sum_{k,m\in\Delta} r_k \underbrace{P_{km} r_m}$$

$$= \langle r, Pr\rangle \qquad P \quad |\Delta|\times|\Delta| \text{- matrix}$$

$$H(r) = \langle r, Pr\rangle - 2\langle q_\Delta, r\rangle + f_0 \sim \min_r$$

Assume that P is invertible.

Look at!

$$\langle P(r-P^{-1}q_\Delta), r-P^{-1}q_\Delta\rangle =$$

$$=\langle Pr, r\rangle - \langle \underbrace{PP^{-1}}q_\Delta, r\rangle$$

$$-\langle Pr, P^{-1}q_\Delta\rangle + \langle \underbrace{PP^{-1}}_{=I}q_\Delta, P^{-1}q_\Delta\rangle$$

$$=\langle Pr, r\rangle - \langle q_\Delta, r\rangle - \langle r, \underbrace{PP^{-1}}_{=I}q_\Delta\rangle$$

$$+\langle q_\Delta, P^{-1}q_\Delta\rangle$$

$$=\langle Pr, r\rangle - 2\langle q_\Delta, r\rangle + \langle q_\Delta, P^{-1}q_\Delta\rangle$$

$$H(r) = \langle P(r-P^{-1}q_\Delta), r-P^{-1}q_\Delta\rangle$$

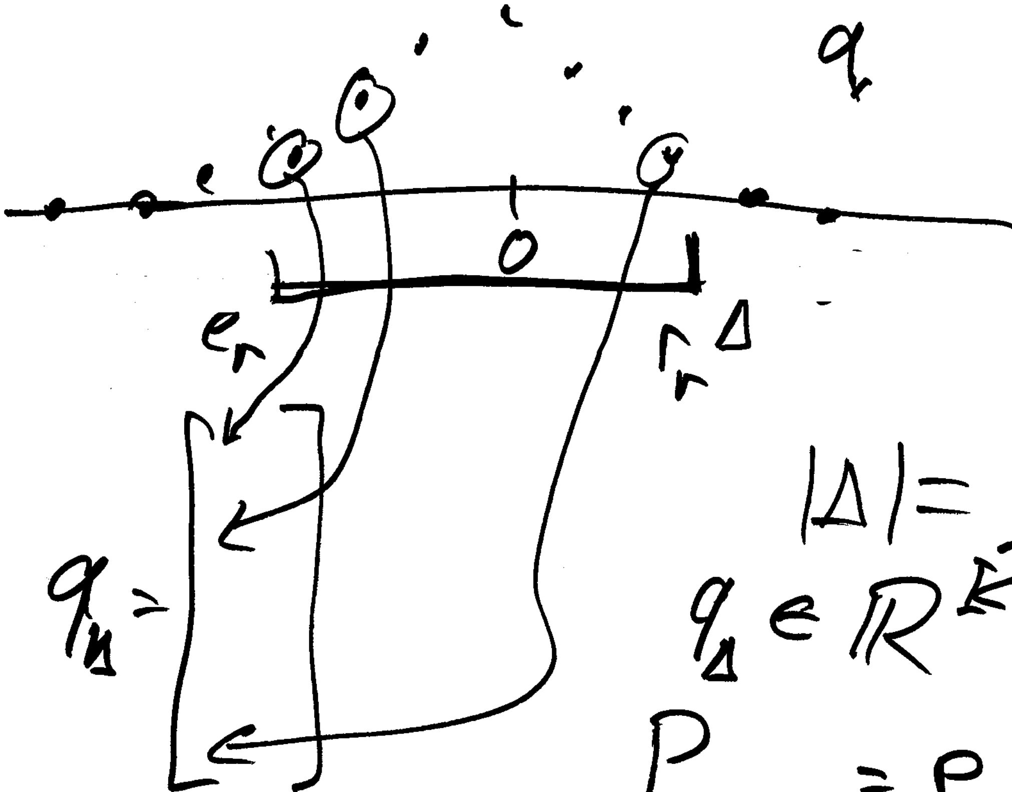
$$+ f_0 - \langle q_\Delta, P^{-1}q_\Delta\rangle$$

P – positive definite

$\Rightarrow H(r) \sim \min$ only when

$r - P^{-1}q_\Delta = 0 \Rightarrow \underline{r = P^{-1}q_\Delta}$

or sol. of $Pr = q_\Delta$



$$\Delta = [\ell_r, r_r]$$

$$|\Delta| = K$$

$$q_\Delta \in \mathbb{R}^K$$

$$\underline{P}_{k,m} = p_{k-m} \qquad K \times K$$

$$\underline{P} = \begin{bmatrix} p_0 & p_1 & & & \\ p_1 & p_0 & \ddots & & \\ p_2 & p_1 & \ddots & \ddots & \\ & & \ddots & \ddots & p_1 \\ & & & p_1 & p_0 \end{bmatrix}$$

Symm Toeplitz matrix

$$\underline{P}_{11} = p_{1-1} = p_0$$

$$\underline{P}_{21} = p_{2-1} = p_1 \qquad \underline{P}_{12} = p_{1-2} = p_{-1} = p_1$$

$$P_{ij} = P_{i-j} = E\, y_i y_j \quad i,j \in \Delta$$

Indeed, $y = a * x + v$

$\operatorname{cov} y = \operatorname{cov}(a*x) + \operatorname{cov}(v)$

$= a * f * a^* + s$

$\Rightarrow P \geq 0$ as a covariance of random vector y_Δ

$$\Rightarrow P \geq 0 \text{ (non neg def)}.$$

$$\left.\begin{matrix} \Rightarrow P \geq 0 \\ + P\text{-invertible} \end{matrix}\right] \Rightarrow P > 0$$

2D field of view

functions on $\mathbb{Z}^2$ - Images.

$$y_i = \sum_{j \in \Delta_a} a_j x_{i-j} = (a*x)_i$$

i - "vector" index

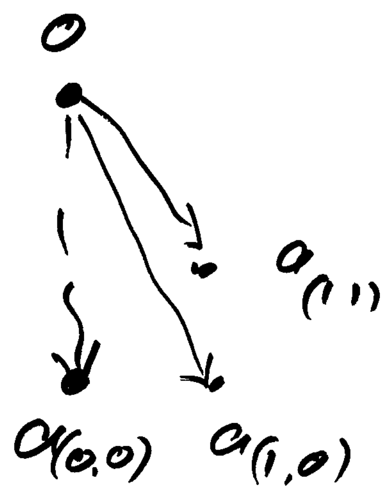
$i = (i_1, i_2) \quad j = (j_1, j_2)$

$i - j = (i_1 - j_1,\ i_2 - j_2)$

$$\delta_i = \begin{cases} 1 & i = 0 \quad (i_1, i_2) = (0\,0) \\ 0 & i \neq 0 \quad (i_1, i_2) \neq (0\,0) \end{cases}$$

$a * \delta$

$$y_i = \sum_j a_j \delta_{i-j} = a_i$$



Point Spread function.

Homogeneous random field

ν - rand function on $\mathbb{Z}^2$

$$\nu_i = \nu_{(i_1, i_2)}$$

$$E\,\nu_i = 0 \qquad E\,\nu_i \nu_j = s_{i-j}$$

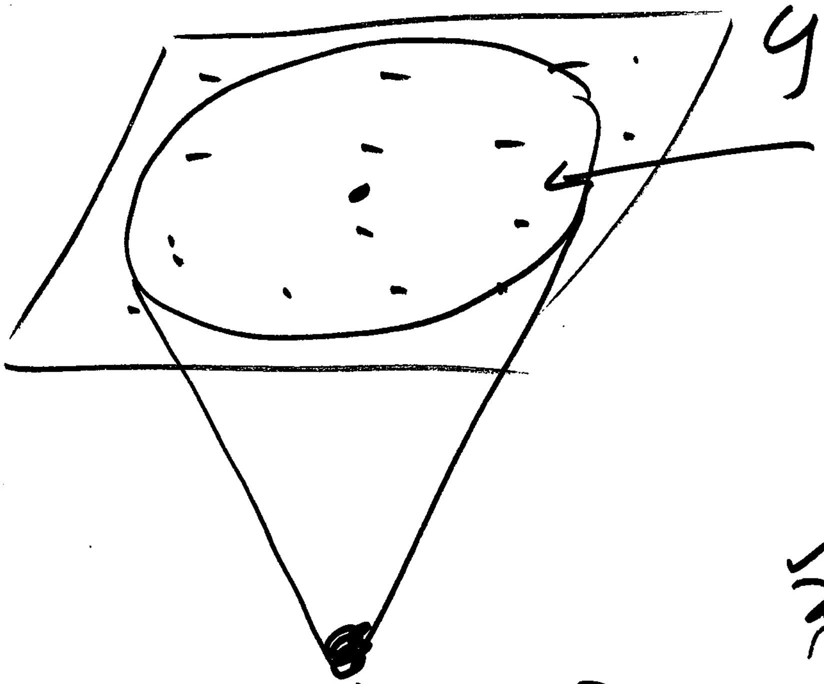
s - function on $\mathbb{Z}^2$

cov ν.

All formulas are the same as for 1D case.

Unknown image x

$y = a * x + v$ in 2D.



r with sapont Δ.

$\hat{x} = r * y$

Find r ! $\Delta_r = \Delta$

and $H(r) \sim \min_{r}$.

$P r = q_{\Delta}$ $\quad |\Delta| = K$ - number of elements in Δ

To find r-optimal:

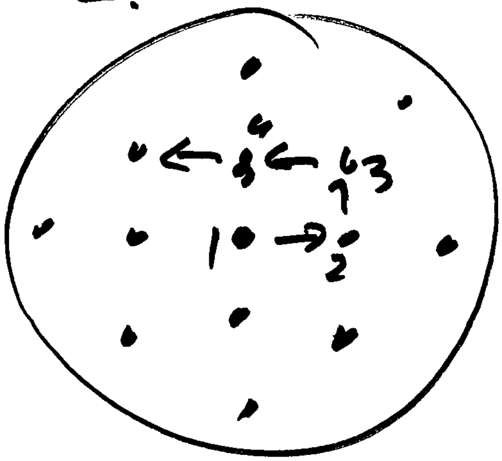
1. $q = f * a^*$

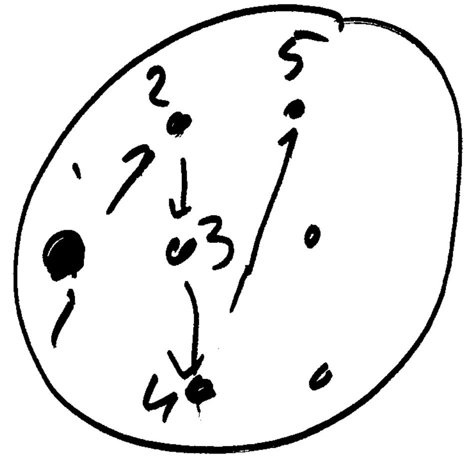
$p = a * f * a^* + S$

$P_{i,j} = p_{i-j}$ $\qquad i = (i_1, i_2) \in \Delta$

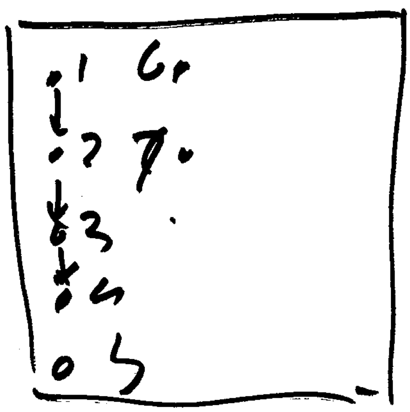
2. enumerate Δ

Δ

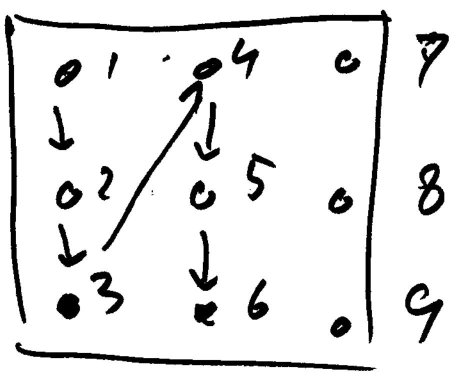


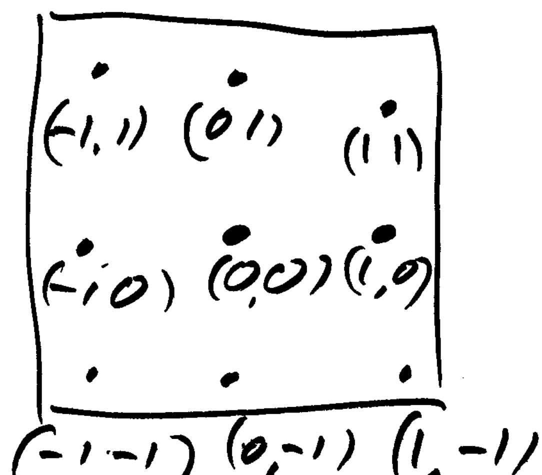


Δ



$\Delta =$





3. $q \to q_\Delta$

$$q_\Delta = \begin{bmatrix} q_{(-1,1)} \\ q_{(-1,0)} \\ \vdots \\ q_{(1,-1)} \end{bmatrix}$$

i - 2D index

$k = 1, \dots K = |\Delta|$

$i_k : i_3 = (-1,-1)$

$K_i : K_{(1,1)} = 7$

$q_{\Delta k} = q_{i_k}$

4. Make $\underline{P}$ $K \times K$ - matrix.

$$\underline{P}_{k\ell} = P_{i_k - i_\ell}$$

$$\underline{P} = \begin{array}{cc} K & \underbrace{\begin{bmatrix} P_{(00)} & P_{01} & & & \\ P_{0,-1} & P_{00} & & & \\ P_{0,-2} & & \ddots & & \\ & & & \ddots & \\ & & & & P_{00} \end{bmatrix}}_{K} \end{array}$$

$\underline{P}_{11} = P_{i_1 - i_1} = P_{(00)}$ $\quad$ $\underline{P}_{33} = P_{i_3 - i_3} = P_{(00)}$

$\underline{P}_{21} = P_{i_2 - i_1} = P_{(-1,0)-(-1,1)} = P_{(0,-1)}$

$\underline{P}_{12} = P_{i_1 - i_2} = P_{(0,1)}$

$P = P^*$ - symmetric function

$P_i = P_{-i}$

$\underline{P}_{31} = P_{i_3 - i_1} = P_{(-1,-1)-(-1,0)} = P_{(0,-2)}$

5. Find $\bar{r} \in \mathbb{R}^{K}$ as a sol of $P\bar{r} = q_{\Delta}$ or $\bar{r} = P^{-1} q_{\Delta}$

6. Transform vector $\bar{r}$ to PSF r

$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_9 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{r}_1 & \bar{r}_4 & \bar{r}_7 \\ \bar{r}_2 & \bar{r}_5 & \bar{r}_8 \\ \bar{r}_3 & \bar{r}_6 & \bar{r}_9 \end{bmatrix}$$

PSF ~~of~~ r

$$r_{i_k} = \bar{r}_k$$

7. $\hat{x} = r * y$

Final Projects

Mon 9

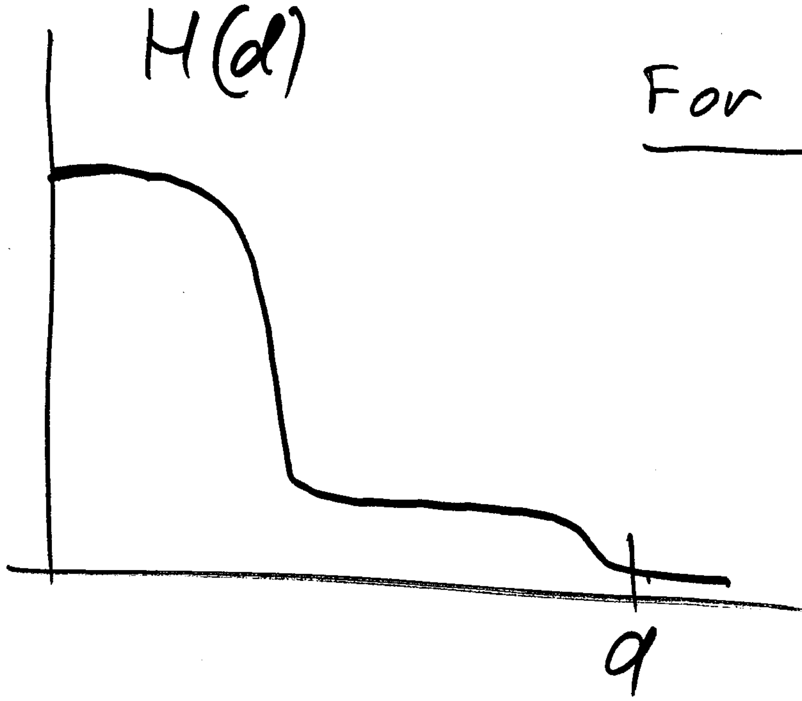
~~Wed 11~~ 1:10 – 3:10

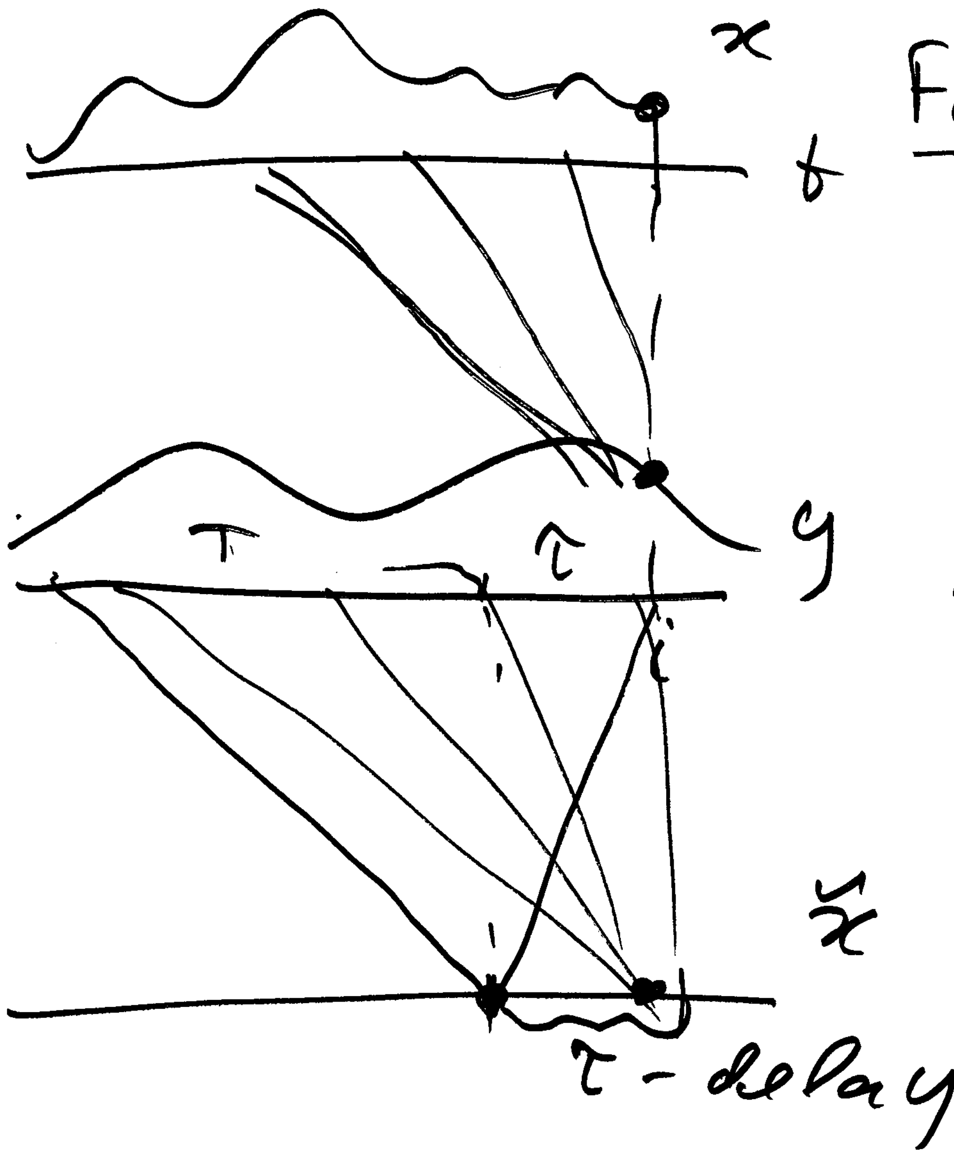
Thur 12

rate projects from 1 (highest) to 4 (Lowest)

No later than May 4.

For Project 2





For Project 3.

$\Delta = [-\tau, T]$