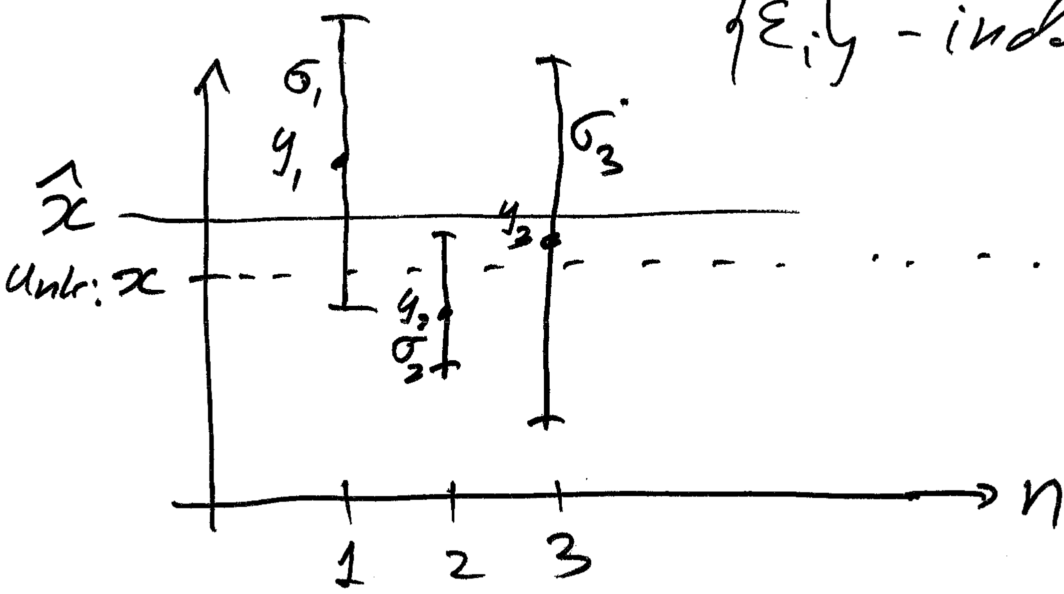
$$y_i = x + \varepsilon_i \qquad i = 1, \ldots n \quad E\varepsilon_i = 0, \; E\varepsilon_i^2 = \sigma_i^2$$

$\{\varepsilon_i\}$ - independent.



(y_1, σ_1^2)

(y_2, σ_2^2)

$\vdots$

(y_n, σ_n^2)

(a) $\hat{x} = \frac{1}{n} \sum_{i=}^{n} y_i$ - unbiased.

$$\hat{x} = \sum_{i=1}^{n} \alpha_i y_i \qquad \alpha_i \text{ - weigts.}$$

$\hat{x}$ - unbiased $\Rightarrow \sum_{i=1}^{n} \alpha_i = 1$

(b) $\alpha_i \sim \frac{1}{\sigma_i} \Rightarrow \sum \frac{c}{\sigma_i} = 1 \Rightarrow c \sum \frac{1}{\sigma_i} = 1$

$\alpha_i = \frac{c}{\sigma_i} \qquad \Rightarrow c = \frac{1}{\sum_i \frac{1}{\sigma_i}}$

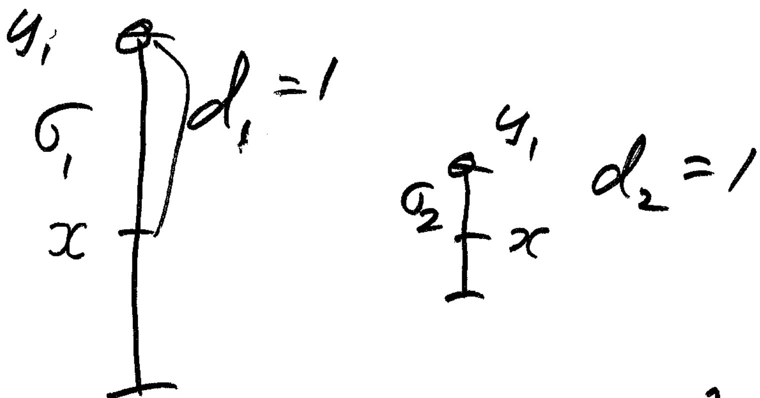
$$\Rightarrow \alpha_i = \frac{1}{\sigma_i \sum_j \frac{1}{\sigma_j}}$$

(c) $\alpha_i \sim \frac{1}{\sigma_i^2}$ $\qquad \alpha_i = \frac{c}{\sigma_i^2}$

$$\sum_i \frac{c}{\sigma_i^2} = 1 \Rightarrow c = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

$$\Rightarrow \alpha_i = \frac{1}{\sigma_i^2 \sum_{j=1}^{n} \frac{1}{\sigma_j^2}}$$

$\frac{|y_i - x|}{\sigma_i} = d_i$ normalized distance from y_i to x



$$Q(x) = \sum_{i=1}^{n} \frac{(y_i - x)^2}{\sigma_i^2} \sim \min_x$$

<u>Weighted Leas Squares</u>

$$\frac{dQ}{dx} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}\left[2x - 2y_i\right] = 0$$

$$\Rightarrow \quad 2x \sum_i \frac{1}{\sigma_i^2} = 2 \sum_i \frac{y_i}{\sigma_i^2}$$

$$\Rightarrow \quad \hat{x} = \frac{\sum_i y_i/\sigma_i^2}{\sum_i \frac{1}{\sigma_i^2}}$$

$$\hat{x}_w = \sum_{i=1}^{n} \underbrace{\frac{1}{\sigma_i^2 \sum_j \frac{1}{\sigma_j^2}}}_{=\alpha_i} y_i$$

$$E(\hat{x} - x)^2 = \qquad \text{since } E\hat{x} = x$$

$$= \operatorname{Var}(\hat{x})$$

$$\hat{x} = \sum_{i=1}^{n} \alpha_i y_i \qquad \sum_i \alpha_i = 1 \ (\text{unb}).$$

$$\operatorname{Var} \hat{x} = E\Big(\sum_i \alpha_i y_i - \underbrace{x}_{=\sum_i \alpha_i x}\Big)^2$$

$$= E\Big[\sum \alpha_i \underbrace{(y_i - x)}_{\varepsilon_i}\Big]^2 = E \sum_i \alpha_i \varepsilon_i \cdot \sum \alpha_j \varepsilon_j$$

$$= \sum_{i,j} \alpha_i \alpha_j \underbrace{E \varepsilon_i \varepsilon_j}_{\text{if } (i \neq j)\ E\varepsilon_i E\varepsilon_j = 0} = \sum_i \alpha_i^2 \sigma_i^2$$

(a) $\alpha_i = \frac{1}{n}$ Compare (a) vs (c)

$$\mathrm{Var}(\hat{x}_u) = \sum \frac{1}{n^2}\sigma_i^2 = \frac{\sum \sigma_i^2}{n^2}$$

(c) $\alpha_i = \dfrac{1}{\sigma_i^2 \sum_j \frac{1}{\sigma_j^2}}$

$$\mathrm{Var}(\hat{x}_w) = \sum_i \frac{1}{\sigma_i^4 \left(\sum_j \frac{1}{\sigma_j^2}\right)^2}\sigma_i^2$$

$$= \sum_i \frac{1}{\sigma_i^2} \cdot \frac{1}{\left(\sum_i \frac{1}{\sigma_i^2}\right)^2} = \frac{1}{\sum_i \frac{1}{\sigma_i^2}}$$

$$\mathrm{Var}(\hat{x}_w) = \frac{1}{\sum_i \frac{1}{\sigma_i^2}}$$

$n = 2$: $\mathrm{Var}(\hat{x}_u) - \mathrm{Var}(\hat{x}_w) =$

$$= \frac{\sigma_1^2 + \sigma_2^2}{4} - \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2} =$$

$$= (\quad) - \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{(\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2\sigma_2^2}{4(\sigma_1^2 + \sigma_2^2)}$$

$$= \frac{(\sigma_1^2 - \sigma_2^2)^2}{4(\sigma_1^2 + \sigma_2^2)} \geq 0$$

"=" only if $\sigma_1^2 = \sigma_2^2$

Optimal Estimation

$$\mathbb{E}(\hat{x}-x)^2 \sim \underset{?}{\min}$$

Good to have unbiased est:

$$\mathbb{E}\,\hat{x} = x$$

$$\hat{x} = \varphi(y_1, y_2 \cdots y_n) \qquad \varphi = ?$$

φ - linear wrt y_i

$$\hat{x} = \varphi = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$$

Unbiased $\Rightarrow$ $\sum_i \alpha_i = 1$

$$H(\alpha_1 \cdots \alpha_n) = \mathbb{E}(\hat{x}-x)^2 = \mathbb{E}\left(\sum \alpha_i y_i - x\right)^2$$

$$= \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2$$

$$\left[\begin{array}{l} H(\alpha_1 \cdots \alpha_n) = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 \sim \underset{\{\alpha_i\}}{\min} \\ \text{condition } \sum_i \alpha_i = 1 \end{array}\right.$$

$$L_1 = H + \lambda\left(\sum_{i=1}^{n} \alpha_i - 1\right)$$

$$= \sum_i \alpha_i^2 \sigma_i^2 + \lambda\left(\sum_i \alpha_i - 1\right) \sim \min_{\alpha_i}$$

$$\frac{\partial L}{\partial \alpha_i} = 2\alpha_i \sigma_i^2 + \lambda = 0$$

$$\Rightarrow \alpha_i = -\frac{\lambda}{2\sigma_i^2}$$

$$\sum_i \alpha_i = 1 \Rightarrow \sum_i \left(-\frac{\lambda}{2\sigma_i^2}\right) = 1$$

$$\lambda \sum \frac{1}{2\sigma_i^2} = -1 \Rightarrow \lambda = -\frac{2}{\sum_i \frac{1}{\sigma_i^2}}$$

$$\alpha_i = \frac{1}{\sigma_i^2 \sum_j \frac{1}{\sigma_j^2}}$$

$$\hat{x} = \sum_i \alpha_i y_i = \underbrace{\boxed{\frac{1}{\sum_i \frac{1}{\sigma_i^2}}}}_{=t} \underbrace{\boxed{\sum_i \frac{y_i}{\sigma_i^2}}}_{=v}$$

$$H(\alpha_{min}) = Var(\hat{x}) = \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

$$t = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \qquad v = \sum_{i=1}^{n} y_i/\sigma_i^2 \qquad \hat{x} = v/t$$

$$\left.\begin{matrix}(y_1, \sigma_1^2)\\ (y_2, \sigma_2^2)\\ \vdots \\ (y_n, \sigma_n^2)\end{matrix}\right] \longmapsto \underline{\underline{(t, v)}} \longmapsto \mathrm{Var}(\hat{x}) = 1/t$$

can Info.

$$\text{Elem: } (y_i, \sigma_i^2) \longmapsto \left(\frac{1}{\sigma_i^2}, \frac{y}{\sigma_i^2}\right)$$

$$\text{Nothing } \emptyset \longmapsto (0, 0)$$

$$\left.\begin{matrix}(y_1, \sigma_1^2)\\ \vdots \\ (y_n, \sigma_n^2)\end{matrix}\right] \longmapsto (v, t)$$

$$\left.\begin{matrix}(\tilde{y}_1, \tilde{\sigma}_1^2)\\ \vdots \\ (\tilde{y}_m, \tilde{\sigma}_m^2)\end{matrix}\right] \longmapsto (\tilde{v}, \tilde{t})$$

$$\oplus \longmapsto (v + \tilde{v}, t + \tilde{t})$$

Linear combination of vectors

$x_1, \dots x_n \in \mathcal{D}$

$\alpha_1 x_1 + \dots + \alpha_n x_n \in \mathcal{D}$

$\alpha_1 \dots \alpha_n$ - coefficients

$x_1 \dots x_n$ - independent if

$\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \Rightarrow \alpha_1 \dots \alpha_n = 0$

$e_1, \dots, e_n$ - Basis in $\mathcal{D}$

if $\forall x \in \mathcal{D} \ \exists! \ \alpha_1 \dots \alpha_n$!

$x = \alpha_1 e_1 + \dots + \alpha_n e_n.$

$\alpha_1 \dots \alpha_n$ - coordinates of x

$\dim \mathcal{D} = n.$

Coord in ONB. $\{e_i\}$ $i=1,..n$

$$x = \sum_{i=1}^{n} x_i e_i$$

$$\langle x, e_j \rangle = \left\langle \sum_i x_i e_i, e_j \right\rangle$$

$$= \sum_i x_i \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = x_j \underbrace{\langle e_j, e_j \rangle}_{=1}$$

$$= x_j$$

Standard example of an n-dim space: $\mathbb{R}^n$

vectors: n-columns:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \qquad \alpha x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

$$\langle x, y \rangle = \sum_i x_i y_i = x^T y, \quad \|x\| = \sqrt{\sum x_i^2}$$

natural Basis:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \text{ - ONB.}$$

$x \in \mathcal{D}$

$$x = \sum_{j=1}^{m} x_j \varphi_j \qquad \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$y = Ax = \sum_{j=1}^{m} x_j A\varphi_j$$

$$\underline{y_i} = \langle e_i, y \rangle = \langle e_i, \sum_{j=1}^{m} A\varphi_j \rangle$$

$$= \sum_j \underbrace{\langle e_i, A\varphi_j \rangle}_{= a_{ij}} x_i = \underline{\sum_j a_{ij} x_j}$$

$$\bar{y} = \bar{A} \bar{x}$$

$\overline{A^*} = \bar{A}^T$ – check!

$$a^*_{ji} = \langle \varphi_j, A^* e_i \rangle = \langle A\varphi_j, e_i \rangle$$

$$= \langle e_i, A\varphi_j \rangle = a_{ij}.$$

$S \geq 0$

$s_{ii} = \langle e_i, S e_i \rangle \geq 0$

if $S > 0$

$s_{ii} > 0$ since $e_i \neq 0$.

$A^* A \geq 0$

$\langle A^* A x, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0$

$\boxtimes$