Глава II. Некоторые классические задачи математической физики

§5. Задача о промерзании (задача о фазовом переходе, задача Стефана)

Построение решения задачи (1) – (4):

$$\begin{cases} u_1 = T_1, \ x = 0, \\ u_2 = T, \ \ t = 0. \end{cases} \qquad (2)$$

$$(2) \Rightarrow u_1 = A_1 + B_1 \Phi\left(\frac{x}{2a_1\sqrt{t}}\right); \ u_2 = A_2 + B_2 \Phi\left(\frac{x}{2a_2\sqrt{t}}\right)$$

$$x = 0: \quad A_1 = T_1; \quad t = 0: \quad A_2 + B_2 = T$$

$$u_1 = u_2 = 0, \ \ x = \xi(t), \ \ \xi(0) = 0 \qquad (3)$$

$$(3) \Rightarrow A_1 + B_1 \Phi\left(\frac{x}{2a_1\sqrt{t}}\right)_{x=\xi} = A_2 + B_2 \Phi\left(\frac{x}{2a_2\sqrt{t}}\right)_{x=\xi} = 0$$

$$A_1 + B_1 \Phi\left(\frac{x}{2a_1\sqrt{t}}\right)_{x=\xi} = 0 \Rightarrow B_1 = -\frac{T_1}{\Phi\left(\dfrac{\xi}{2a_1\sqrt{t}}\right)}$$

$$A_2 = T - B_2 = -B_2 \Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right) \Rightarrow$$

$$T = B_2\left(1 - \Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)\right) \Rightarrow B_2 = \frac{T}{1 - \Phi\left(\dfrac{\xi}{2a_2\sqrt{t}}\right)}$$

$$A_2 = T - B_2 = T - \frac{T}{1 - \Phi\left(\dfrac{\xi}{2a_2\sqrt{t}}\right)} = -\frac{T\Phi\left(\dfrac{\xi}{2a_2\sqrt{t}}\right)}{1 - \Phi\left(\dfrac{\xi}{2a_2\sqrt{t}}\right)}$$

$$k_1 \frac{\partial u_1}{\partial x}\bigg|_{x=\xi} - k_2 \frac{\partial u_2}{\partial x}\bigg|_{x=\xi} = \lambda\rho \frac{d\xi}{dt} \qquad (4)$$

$$(4) \Rightarrow \frac{\partial u_1}{\partial x} = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)} \frac{\partial}{\partial x} \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2a_1\sqrt{t}}} e^{-z^2} dz = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)} \frac{2}{\sqrt{\pi}} \frac{1}{2a_1\sqrt{t}} e^{-\frac{x^2}{4a_1^2 t}}$$

$$\frac{\partial u_2}{\partial x} = -\frac{T}{1-\Phi\left(\frac{\alpha}{2a_2}\right)} \frac{2}{\sqrt{\pi}} \frac{1}{2a_2\sqrt{t}} e^{-\frac{x^2}{4a_2^2 t}}$$

$$-k_1 \frac{T_1}{a_1\sqrt{\pi t}\,\Phi\left(\frac{\alpha}{2a_1}\right)} e^{-\frac{\xi^2}{4a_1^2 t}} - k_2 \frac{T}{a_2\sqrt{\pi t}\,\Phi\left(1-\Phi\left(\frac{\alpha}{2a_2}\right)\right)} e^{-\frac{\xi^2}{4a_2^2 t}} =$$

$$= \lambda\rho \frac{d}{dt}\left(\alpha\sqrt{t}\right) = \frac{\lambda\rho\alpha}{2\sqrt{t}}, \quad \text{где } \alpha = \frac{\xi}{\sqrt{t}} \quad \Rightarrow$$

$$\frac{k_1 T_1 e^{-\frac{\alpha^2}{4a_1^2}}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)} + \frac{k_2 T_1 e^{-\frac{\alpha^2}{4a_2^2}}}{a_2\left(1-\Phi\left(\frac{\alpha}{2a_2}\right)\right)} = -\lambda\rho\alpha\frac{\sqrt{\pi}}{2}$$

$$\text{При } T=0:\ B_2 = \frac{T}{1-\Phi\left(\frac{\alpha}{2a_2}\right)} = 0 \Rightarrow A_2 = T - B_2 = 0:$$

$$A_1 = T_1;\ B_1 = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)};\ A_2 = 0;\ B_2 = 0$$

$$\frac{k_1 T_1 e^{-\frac{\alpha^2}{4a_1^2}}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)} = -\lambda\rho\alpha\frac{\sqrt{\pi}}{2} \qquad (10)$$

Положим $\beta = \dfrac{\alpha}{2a_1}$, $D = \dfrac{\lambda \rho a_1^2}{k_1 T_1}$

Тогда из (10) $\Rightarrow \dfrac{1}{\sqrt{\pi}} \dfrac{e^{-\beta^2}}{\Phi(\beta)} = -D\beta$