

Глава II. Некоторые классические задачи математической физики

§4. Динамика сорбции газа

Для проверки условия $u(0, t) = u_0$ (15) сделаем во втором слагаемом правой части формулы (20) замену $\theta = \frac{\tau}{x_1}$. Тогда получим

Положим теперь $x_1 = 0$:

$$u = u_0 \left\{ e^{-t_1} + \int_0^{t_1} e^{-\theta} d\theta \right\} = u_0 \{ e^{-t_1} - e^{-t_1} + 1 \} = u_0$$

Выполнение условия (17) очевидно:

$$u(x, 0) = u_0 e^{-x_1} = u_0 e^{-\frac{\beta}{v}x}$$

$$t' = t - \frac{x}{v}, \quad x' = x; \quad t'_0 = t_0 - \frac{x}{v} = 0 \Rightarrow t_0 = \frac{x}{v}$$

$$-v \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial a}{\partial t} \quad (7); \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} + \frac{\partial u}{\partial t'} \frac{\partial t'}{\partial x} \Rightarrow$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} - \frac{1}{v} \frac{\partial u}{\partial t'}; \quad (7) \Rightarrow -v \frac{\partial u}{\partial x'} + \frac{\partial u}{\partial t'} = \frac{\partial u}{\partial t'} + \frac{\partial a}{\partial t'} \Rightarrow -v \frac{\partial u}{\partial x'} = \frac{\partial a}{\partial t'} \quad (22)$$

$$\frac{\partial a}{\partial t} = \beta(u - \gamma a) \quad (8); \quad \frac{\partial a}{\partial t} = \frac{\partial a}{\partial t'}; \quad (8) \Rightarrow \frac{\partial a}{\partial t'} = \beta(u - \gamma a) \quad (23)$$

$$\xi = \frac{x' \beta}{v}, \quad \tau = t' \beta \quad (24); \quad \frac{\partial u}{\partial x'} = \frac{\partial u}{\partial \xi} \frac{\beta}{v}; \quad \frac{\partial a}{\partial t'} = \frac{\partial a}{\partial \tau} \beta$$

$$(22) \Rightarrow -v \frac{\partial u}{\partial \xi} \frac{\beta}{v} = \frac{\partial a}{\partial \tau} \beta \Rightarrow -\frac{\partial u}{\partial \xi} = \frac{\partial a}{\partial \tau} \Rightarrow \frac{\partial u}{\partial \xi} + \frac{\partial a}{\partial \tau} = 0 \quad (25)$$

$$(23) \Rightarrow \frac{\partial a}{\partial \tau} \beta = \beta(u - \gamma a) \Rightarrow \frac{\partial a}{\partial \tau} = u - \gamma a \quad (26)$$

$$t = t' + \frac{x'}{v} = \frac{1}{\beta} \tau + \frac{1}{\beta} \xi = 0 \Rightarrow \tau + \xi = 0, \tau = -\xi$$

$$a(x, 0) = 0 \quad (9) \Rightarrow a|_{\tau=-\xi} = 0 \quad (27)$$

$$u(x, 0) = 0 \quad (10) \Rightarrow u|_{\tau=-\xi} = 0 \quad (28)$$

$$u(0, t) = u_0 \quad (11) \Rightarrow u|_{\xi=0} = u_0 \quad (29)$$

$$t' = t - \frac{x}{v}, \quad x' = x; \quad \xi = \frac{x'\beta}{v}, \quad \tau = t'\beta,$$

$$(25) \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \tau} = -\frac{\partial^2 a}{\partial \tau^2}; \quad (26) \Rightarrow \frac{\partial^2 a}{\partial \tau^2} = \frac{\partial u}{\partial \tau} - \gamma \frac{\partial a}{\partial \tau}$$

$$\frac{\partial^2 u}{\partial \xi \partial \tau} = -\frac{\partial u}{\partial \tau} + \gamma \frac{\partial a}{\partial \tau}; \quad (25) \Rightarrow \frac{\partial a}{\partial \tau} = -\frac{\partial u}{\partial \xi} \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \tau} + \frac{\partial u}{\partial \tau} + \gamma \frac{\partial u}{\partial \xi} = 0 \quad (30)$$

$$(7)-(10) \Rightarrow (10) \Rightarrow u(x, 0) = 0; \quad (8) \Rightarrow \frac{\partial a}{\partial t}(x, 0) = \beta(u(x, 0) - \gamma a(x, 0)) = 0$$

$$(10) \Rightarrow \frac{\partial u}{\partial x}(x, 0) = 0; \quad (7) \Rightarrow \frac{\partial u}{\partial t}(x, 0) = -v \frac{\partial u}{\partial x}(x, 0) - \frac{\partial a}{\partial t}(x, 0) = 0;$$

$$u(x, 0) = 0; \quad \frac{\partial u}{\partial t}(x, 0) = 0; \quad x = \frac{v}{\beta} \xi; \quad t = t' + \frac{x}{v} = \frac{1}{\beta} \tau + \frac{1}{\beta} \xi;$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial \xi} = \frac{v}{\beta} \frac{\partial u}{\partial x} + \frac{1}{\beta} \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial \xi} \Big|_{\tau=-\xi} = \frac{v}{\beta} \frac{\partial u}{\partial x}(x, 0) + \frac{1}{\beta} \frac{\partial u}{\partial t}(x, 0) = 0$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t} \frac{\partial t}{\partial \tau} = \frac{1}{\beta} \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial \tau} \Big|_{\tau=-\xi} = \frac{1}{\beta} \frac{\partial u}{\partial t}(x, 0) = 0$$

$$\frac{\partial u}{\partial n} \Big|_{\tau=-\xi} = \left(\frac{\partial u}{\partial \tau} \cos(\hat{n}, \tau) + \frac{\partial u}{\partial \xi} \cos(\hat{n}, \xi) \right) \Big|_{\tau=-\xi} = 0$$

$$u \Big|_{\tau=-\xi} = u(x, 0) = 0$$

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial \xi \partial \tau} + \frac{\partial u}{\partial \tau} + \gamma \frac{\partial u}{\partial \xi} = 0, \quad (\xi, \tau) \in D \quad (30) \\ u \Big|_{\tau=-\xi} = 0 \Rightarrow u = 0, \quad (\xi, \tau) \in D \\ \frac{\partial u}{\partial n} \Big|_{\tau=-\xi} = 0, \quad (\xi, \tau) \in D \end{array} \right.$$

$$\frac{\partial a}{\partial t} = -v \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \Rightarrow a = \frac{\beta}{\gamma} u - \frac{1}{\gamma} \frac{\partial a}{\partial t} = 0$$

$$\xi = \frac{\beta}{v}x, \quad \theta = \beta\gamma t \quad (35) \Rightarrow \frac{\partial^2 u}{\partial x \partial t} = \frac{\beta}{v} \beta\gamma \frac{\partial^2 u}{\partial \xi \partial \theta}; \quad \frac{\beta}{v} \frac{\partial u}{\partial t} = \frac{\beta}{v} \beta\gamma \frac{\partial u}{\partial \theta}; \quad \beta\gamma \frac{\partial u}{\partial x} = \beta\gamma \frac{\beta}{v} \frac{\partial u}{\partial \xi};$$

$$u_{xt} + \frac{\beta}{v}u_t + \beta\gamma u_x = 0 \quad (16) \Rightarrow \frac{\beta^2}{v} \gamma u_{\xi\theta} + \frac{\beta}{v} \beta\gamma u_\theta + \beta\gamma \frac{\beta}{v} u_\xi = 0 \Rightarrow u_{\xi\theta} + u_\theta + u_\xi = 0 \quad (36)$$

$$u(\xi, \theta) = w(\xi, \theta)e^{-\xi-\theta} \quad (39) \Rightarrow$$

$$\frac{\partial u}{\partial \theta} = \left(\frac{\partial w}{\partial \theta} - w\right)e^{-\xi-\theta}; \quad \frac{\partial u}{\partial \xi} = \left(\frac{\partial w}{\partial \xi} - w\right)e^{-\xi-\theta}$$

$$\frac{\partial^2 u}{\partial \xi \partial \theta} = \left(\frac{\partial^2 w}{\partial \xi \partial \theta} - \frac{\partial w}{\partial \theta} - \frac{\partial w}{\partial \xi} + w\right)e^{-\xi-\theta} \Rightarrow$$

$$w_{\xi\theta} - w = 0 \quad (40)$$

$$u(x, 0) = u_0 e^{-\frac{\beta}{v}x} \Rightarrow u(\xi, 0) = e^{-\xi} \quad (37)$$

$$\text{при } u_0 = 1: \quad u(0, \theta) = 1 \quad (38); \quad (37), (38) \Rightarrow$$

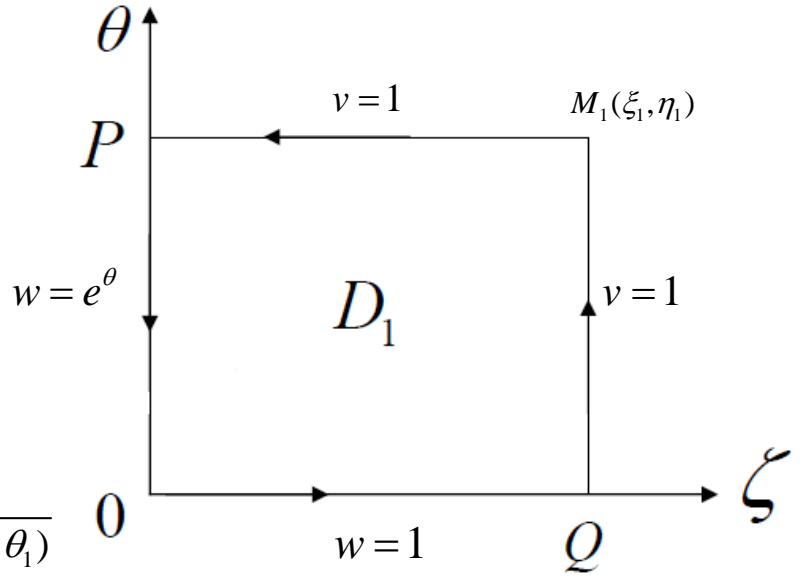
$$u(\xi, 0) = w(\xi, 0)e^{-\xi} = e^{-\xi} \Rightarrow w(\xi, 0) = 1 \quad (41)$$

$$u(0, \theta) = w(0, \theta)e^{-\theta} = 1 \Rightarrow w(0, \theta) = e^\theta \quad (42)$$

Функция Римана для уравнения (40):

$$v(\xi, \theta, \xi_1, \theta_1) = I_0(2\sqrt{(\xi - \xi_1)(\theta - \theta_1)}) \quad (43)$$

$$\begin{cases} w_{\xi\theta} - w = 0 \\ w(\xi, 0) = 1 \\ w(0, \theta) = e^\theta \end{cases}$$



$$v(\xi, \theta, \xi_1, \theta_1) = I_0(2\sqrt{(\xi - \xi_1)(\theta - \theta_1)})$$

$$\begin{cases} v_{\xi\theta} - v = 0 & \text{в } D_1 \quad (45) \end{cases}$$

$$\begin{cases} v = 1 & \text{на } PM_1 \text{ и } M_1Q \quad (46) \end{cases}$$

$$\int_{D_1} (wv_{\xi\theta} - w_{\xi\theta}v) d\xi d\theta = \int_{D_1} (wv - vw) d\xi d\theta = 0 \Rightarrow$$

$$\int_0^Q \underbrace{(v_\xi w - vw_\xi)}_{\text{I}} d\xi + \int_Q^{M_1} \underbrace{(vw_\theta - v_\theta w)}_{\text{II}} d\theta + \int_{M_1}^P \underbrace{(v_\xi w - vw_\xi)}_{\text{III}} d\xi + \int_P^0 \underbrace{(vw_\theta - v_\theta w)}_{\text{IV}} d\theta = 0$$

$$\text{I) } \int_0^Q v_\xi w d\xi = \int_0^Q v_\xi d\xi = v \Big|_0^Q = v(Q) - v(0) = I_0(0) - I_0(2\sqrt{\xi_1\theta_1}) = 1 - I_0(2\sqrt{\xi_1\theta_1}) \quad (49)$$

$$\text{II) } \int_Q^{M_1} vw_\theta d\theta = \int_Q^{M_1} w_\theta d\theta = v \Big|_Q^{M_1} = w(M_1) - w(Q) = w(M_1) - 1 \quad (50)$$

$$\text{III) } - \int_{M_1}^P vw_\xi d\xi = - \int_{M_1}^P w_\xi d\xi = w(M_1) - w(P) = w(M_1) - e^{\theta_1} \quad (51)$$

$$\begin{aligned}
\text{IV) } \int_P^0 e^\theta (v - v_\theta) d\theta &= \int_P^0 e^\theta v d\theta - \int_P^0 e^\theta v_\theta d\theta = \int_P^0 e^\theta v d\theta - e^\theta v \Big|_P^0 + \int_P^0 e^\theta v d\theta = \\
&= (e^\theta v)(P) - (e^\theta v)(0) + 2 \int_P^0 e^\theta v d\theta = e^{\theta_1} - I_0(2\sqrt{\xi_1 \theta_1}) + 2 \int_P^0 e^\theta I_0(2\sqrt{(\xi - \xi_1)(\theta - \theta_1)}) d\theta \quad (52)
\end{aligned}$$

$$\begin{aligned}
I - \text{IV} \Rightarrow 0 &= \lambda - I_0(2\sqrt{\xi_1 \theta_1}) + w(M_1) - \lambda + w(M_1) - e^{\theta_1} + e^{\theta_1} - I_0(2\sqrt{\xi_1 \theta_1}) + \\
&+ 2 \int_P^0 e^\theta I_0(2\sqrt{(\xi - \xi_1)(\theta - \theta_1)}) d\theta \Rightarrow
\end{aligned}$$

$$w(M_1) = I_0(2\sqrt{\xi_1 \theta_1}) - \int_{\theta_1}^0 e^\theta I_0(2\sqrt{(0 - \xi_1)(\theta - \theta_1)}) d\theta \Rightarrow$$

$$w(M_1) = I_0(2\sqrt{\xi_1 \theta_1}) - \int_{\theta_1}^0 e^\theta I_0(2\sqrt{\xi_1(\theta_1 - \theta)}) d\theta = I_0(2\sqrt{\xi_1 \theta_1}) + \int_0^{\theta_1} e^\theta I_0(2\sqrt{\xi_1(\theta_1 - \theta)}) d\theta \quad (53)$$

$$(39) \Rightarrow u(M_1) = u(\xi_1, \theta_1) = e^{-\xi_1 - \theta_1} w(M_1) \quad (54)$$

$$(53), (54) \Rightarrow u(\xi_1, \theta_1) = e^{-\xi_1} \left\{ e^{-\theta_1} I_0(2\sqrt{\xi_1 \theta_1}) + \int_0^{\theta_1} e^{\theta - \theta_1} I_0(2\sqrt{\xi_1(\theta_1 - \theta)}) d\theta \right\} \quad (55)$$

$$\begin{aligned} \text{Çàì àú à: } \tau = \xi_1(\theta_1 - \theta) &\Rightarrow \theta_1 - \theta = \frac{\tau}{\xi_1} \Rightarrow d\theta = -\frac{1}{\xi_1} d\tau \Rightarrow \\ \int_0^{\theta_1} e^{\theta - \theta_1} I_0(2\sqrt{\xi_1(\theta_1 - \theta)}) d\theta &= -\frac{1}{\xi_1} \int_{\xi_1 \theta_1}^0 e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau}) d\tau = \frac{1}{\xi_1} \int_0^{\xi_1 \theta_1} e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau}) d\tau \Rightarrow \\ u(\xi_1, \theta_1) &= e^{-\xi_1} \left\{ e^{-\theta_1} I_0(2\sqrt{\xi_1 \theta_1}) + \frac{1}{\xi_1} \int_0^{\xi_1 \theta_1} e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau}) d\tau \right\} \quad (56) \end{aligned}$$

В формуле (20): $\xi_1 = x_1, \theta_1 = t_1$ поэтому формула (56) совпадает с формулой (20) при $u_0 = 1$