§2. Задача Гурса. Мажорантные оценки

Формулы (13)-(15). Из формулы (13) получаем:

$$z_1 = -\int_0^y\int_0^x \{a\frac{\partial z_0}{\partial \xi} + b\frac{\partial z_0}{\partial \eta} + cz_0\} d\xi d\eta \;\; \Rightarrow |z_1| \le 3MHxy \le 3MH\frac{(x+y)^2}{2!}$$

$$\frac{\partial z_1}{\partial x} = -\int_0^y \{a\frac{\partial z_0}{\partial x} + b\frac{\partial z_0}{\partial \eta} + cz_0\} d\eta \;\Rightarrow \left|\frac{\partial z_1}{\partial x}\right| \le 3MHy \le 3MH(x+y)$$

Аналогично:

$$\left|\frac{\partial z_1}{\partial y}\right| \le 3MHx \le 3MH(x+y)$$

Индукция:

$$z_n = -\int_0^y\int_0^x \{a\frac{\partial z_{n-1}}{\partial \xi} + b\frac{\partial z_{n-1}}{\partial \eta} + cz_{n-1}\} d\xi d\eta$$

$$\left|z_{n-1}\right| \le 3HM^{n-1}K^{n-2}\frac{(x+y)^n}{n!}$$

$$\left|\frac{\partial z_{n-1}}{\partial x}\right| \le 3HM^{n-1}K^{n-2}\frac{(x+y)^{n-1}}{(n-1)!}$$

$$\left|\frac{\partial z_{n-1}}{\partial y}\right| \le 3HM^{n-1}K^{n-2}\frac{(x+y)^{n-1}}{(n-1)!},$$

где K = L + 2.

$$\left|z_n\right| \le 3HM^nK^{n-2}\left\{\frac{2}{(n-1)!}\int_0^y\int_0^x (\xi+\eta)^{n-1} d\xi d\eta + \frac{1}{n!}\int_0^y\int_0^x (\xi+\eta)^n d\xi d\eta\right\},$$

$$\int_0^y\int_0^x (\xi+\eta)^{n-1} d\xi d\eta \le \frac{1}{n}\int_0^y (x+\eta)^n d\eta \le \frac{1}{n(n+1)}(x+y)^{n+1},$$

$$\int_0^y\int_0^x (\xi+\eta)^n d\xi d\eta \le \frac{1}{(n+1)(n+2)}(x+y)^{n+2},$$

$$\{\ldots\} \le \frac{2}{(n-1)!}\frac{1}{n(n+1)}(x+y)^{n+1} + \frac{1}{n!}\frac{1}{(n+1)(n+2)}(x+y)^{n+2} =$$

$$= \frac{2}{(n+1)!}(x+y)^{n+1} + \frac{1}{(n+2)!}(x+y)^{n+2} = \frac{(x+y)^{n+1}}{(n+1)!}(2+\frac{x+y}{n+2}) <$$

$$< \frac{(x+y)^{n+1}}{(n+1)!}(2+\frac{2L}{n+2}) < \frac{(x+y)^{n+1}}{(n+1)!}(2+L) = \frac{(x+y)^{n+1}}{(n+1)!}K \Rightarrow$$

$$\left|z_n\right| \le 3HM^n K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}.$$

$$\frac{\partial z_n}{\partial x} = -\int_0^y \{a\frac{\partial z_{n-1}}{\partial x} + b\frac{\partial z_{n-1}}{\partial \eta} + cz_{n-1}\}d\eta,$$

$$\left|\frac{\partial z_n}{\partial x}\right| \le 3HM^n K^{n-2}\left\{\frac{2}{(n-1)!}\int_0^y (x+\eta)^{n-1}d\eta + \frac{1}{n!}\int_0^y (x+\eta)^n d\eta\right\},$$

$$\{\dots\} = \frac{2}{(n-1)!}\frac{(x+y)^n}{n} + \frac{1}{n!}\frac{(x+y)^{n+1}}{n+1} = 2\frac{(x+y)^n}{n!} + \frac{(x+y)^{n+1}}{(n+1)!},$$

$$\left|\frac{\partial z_n}{\partial x}\right| \le 3HM^n K^{n-2}\frac{(x+y)^n}{n!}(2+\overset{<\,2L}{\frac{(x+y)}{n+1}}) =$$

$$= 3HM^n K^{n-2}\frac{(x+y)^n}{n!}\overset{<\,2+L=K}{(2+\frac{2}{n+1}L)} < 3HM^n K^{n-1}\frac{(x+y)^n}{n!}.$$

Аналогично:

$$\left|\frac{\partial z_n}{\partial y}\right| < 3HM^n K^{n-1}\frac{(x+y)^n}{n!}$$

$$\left|z_n\right| \le 3HM^n K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!} < 3HM^n K^{n-1}\frac{(2L)^{n+1}}{(n+1)!} = \frac{3H}{K^2 M}\frac{(2KLM)^{n+1}}{(n+1)!}$$

$$\left|\frac{\partial z_n}{\partial x}\right| \le 3HM^n K^{n-1}\frac{(x+y)^n}{n!} < 3HM^n K^{n-1}\frac{(2L)^n}{n!} = \frac{3H}{K}\frac{(2KLM)^n}{n!}$$