Глава 2. Некоторые классические задачи математической физики

§1. Уравнение $\triangle u + cu = -f$ в неограниченной области

Парциальные условия излучения ($e^{-i\omega t}$)

$$\int_0^b \left\{ \frac{\partial u}{\partial x} + i\gamma_n^{(1)} u \right\}_{x=0} \psi_n(y)\,dy = 2i\gamma_{n_0} A_{n_0} \delta_{nn_0} \quad (44) \quad (n = 1, 2, \ldots)$$

Падающая(правая) волна

$$u_{n_0}(x, y) = A_{n_0} e^{i\gamma_{n_0} x} \psi_{n_0}(y) \quad \Rightarrow$$

$$\int_0^b \left\{ \frac{\partial u_{n_0}}{\partial x} + i\gamma_n^{(1)} u_0 \right\}_{x=0} \psi_n(y)\,dy = \int_0^b \left\{ i\gamma_{n_0}^{(1)} A_{n_0} + i\gamma_n^{(1)} A_{n_0} \right\}_{x=0} \psi_{n_0}(y)\psi_n(y)\,dy =$$

$$= \left\{ i\gamma_{n_0}^{(1)} A_{n_0} + i\gamma_n^{(1)} A_{n_0} \right\} \boxed{\int_0^b \psi_{n_0}(y)\psi_n(y)\,dy} = \\ {}_{= \delta_{nn_0}}$$

$$= 2i\gamma_{n_0} A_{n_0} \delta_{nn_0} \quad \Rightarrow \text{ падающая волна } u_{n_0}(x, y) \text{ проходит сечение x=0}$$

Любая правая волна при $n \neq n_0$ пройти сечение x=0 не может:

$$\overset{(\rightarrow)}{u_k}(x, y) = A_k e^{i\gamma_k x} \psi_k(y), \quad k \neq n_0 \Rightarrow$$

$$\int_0^b \left\{ \frac{\partial \overset{(\rightarrow)}{u_k}}{\partial x} + i\gamma_n^{(1)} \overset{(\rightarrow)}{u_k} \right\}_{x=0} \psi_n(y)\,dy = \left\{ i\gamma_k^{(1)} A_k + i\gamma_n^{(1)} A_k \right\}_{x=0} \boxed{\int_0^b \psi_n(y)\psi_k(y)\,dy} = \\ {}_{= \delta_{nk}}$$

$$= 2i\gamma_k^{(1)} A_k = 2i\gamma_{n_0}^{(1)} A_n \boxed{\delta_{kn_0}}^{=0} = 0 \quad \Rightarrow \quad A_k = 0, k \neq n_0$$

Любая левая волна проходит сечение x=0: $\overset{(\leftarrow)}{u_k}(x,y) = B_k e^{-i\gamma_k x}\psi_k(y) \quad \Rightarrow$

$$\int_0^b \left\{ \frac{\partial \overset{(\leftarrow)}{u_k}}{\partial x} + i\gamma_n^{(1)} \overset{(\leftarrow)}{u_k} \right\}_{x=0} \psi_n(y)dy = \overset{=\delta_{nk}}{}$$

$$= \left\{ -i\gamma_k^{(1)}B_k + i\gamma_n^{(1)}B_n \right\}_{x=0} \boxed{\int_0^b \psi_n(y)\psi_k(y)dy} =$$

$$= (-i\gamma_n^{(1)}B_n + i\gamma_n^{(1)}B_n) = 0$$

Квадрупольный излучатель

$$p(r,\theta) = \sum_{n=0}^{\infty} C_n \zeta_n^{(1)}(kr)P_n(\cos\theta) \qquad (61)$$

$$\zeta_n^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}} H_{n+\frac{1}{2}}^{(1)}(kr) \qquad (62)$$

$$\frac{\partial p}{\partial r}\Big|_{r=a} = i\omega\rho_0 v_0 P_2(\cos\theta) \quad (58) \quad \Rightarrow$$

$$\sum_{n=0}^{\infty} C_n k \zeta_n^{(1)\prime}(ka)P_n(\cos\theta) = i\omega\rho_0 v_0 P_2(\cos\theta)$$

$$C_2 k \zeta_2^{(1)\prime}(ka) = i\omega\rho_0 v_0 \quad \Rightarrow$$

$$C_2 = i\overset{=c}{\left(\frac{\omega}{k}\right)}\frac{\rho_0 v_0}{\zeta_2^{(1)\prime}(ka)} = ic\rho_0 v_0 \frac{1}{\zeta_2^{(1)\prime}(ka)} \quad \Rightarrow$$

$$p(r,\theta) = ic\rho_0 v_0 \frac{\zeta_2^{(1)}(kr)}{\zeta_2^{(1)\prime}(ka)}P_2(\cos\theta) \qquad (64)$$

$$v_r = \frac{1}{i\omega\rho_0}\frac{\partial p}{\partial r} \qquad (55) \Rightarrow$$

$$\overset{=\omega}{}$$

$$v_r(r,\theta) = \frac{1}{i\omega\rho_0} \, ic\rho_0 v_0 k \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) =$$

$$= v_0 \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) \qquad (65)$$

$$ka = \frac{2\pi}{\lambda} a \ll 1, \quad kr = \frac{2\pi}{\lambda} r \gg 1$$

$$p(r,\theta) = ic\rho_0 v_0 \frac{\zeta_2^{(1)}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) = ic\rho_0 v_0 \frac{-\dfrac{1}{kr} e^{ikr} e^{-i\frac{\pi}{2}}}{i\dfrac{9}{(ka)^4}} P_2(\cos\theta) =$$

$$= \frac{ic\rho_0 v_0 k^4 a^4}{i9kr} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) = -\frac{c\rho_0 v_0 k^3 a^4}{9r} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) \quad (67)$$

$$v_r(r,\theta) = v_0 \frac{\zeta_2^{(1)\prime}(kr)}{\zeta_2^{(1)\prime}(ka)} P_2(\cos\theta) = v_0 \frac{-\dfrac{i}{kr} e^{ikr} e^{-i\frac{\pi}{2}}}{i\dfrac{9}{(ka)^4}} P_2(\cos\theta) =$$

$$= -\frac{v_0 k^4 a^4}{9kr} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) = -\frac{v_0 k^3 a^4}{9r} e^{ikr} e^{-i\frac{\pi}{2}} P_2(\cos\theta) \quad (68)$$

$$\overline{P}(r,\theta,t) = \mathrm{Re}(p(r,\theta)e^{-i\omega t}) = \mathrm{Re}\left(-\frac{c\rho_0 v_0 k^3 a^4}{9r} P_2(\cos\theta) e^{-i\left(\omega t - kr + \frac{\pi}{2}\right)} \right) =$$

$$= -\frac{c\rho_0 v_0 k^3 a^4}{9r} P_2(\cos\theta)\cos\left(\omega t - kr + \frac{\pi}{2}\right)$$

$$\overline{V}_r\left(r,\theta,t\right) = \text{Re}(v_r(r,\theta)e^{-i\omega t}) = \text{Re}\left(-\frac{v_0 k^3 a^4}{9r} P_2(\cos\theta)e^{-i\left(\omega t - kr + \frac{\pi}{2}\right)}\right) =$$

$$= -\frac{v_0 k^3 a^4}{9r} P_2(\cos\theta)\cos\left(\omega t - kr + \frac{\pi}{2}\right)$$

$$\overline{Y} = \frac{1}{T}\int_0^T \overline{P}(r,\theta,t)\overline{V}_r\left(r,\theta,t\right)dt = \frac{c\rho_0 v_0^2 k^6 a^8 P_2^2(\cos\theta)}{81 r^2}\frac{1}{T}\int_0^T \cos^2\left(\omega t - kr + \frac{\pi}{2}\right)dt =$$

$$\frac{1}{T}\int_0^T \cos^2\left(\omega t - kr + \frac{\pi}{2}\right)dt = \frac{1}{2T}\int_0^T (1 + \cos 2\left(\omega t - kr + \frac{\pi}{2}\right))dt = \frac{1}{2} \Rightarrow$$

$$\overline{Y} = \frac{c\rho_0 v_0^2 k^6 a^8 P_2^2(\cos\theta)}{162 r^2}$$

$$\Pi = r^2 \int_0^{2\pi}\int_0^\pi \overline{Y}\sin\theta\, d\theta\, d\varphi = \frac{2\pi c\rho_0 v_0^2 k^6 a^8}{162}\int_\pi^0 P_2^2(\cos\theta)d\cos\theta =$$

$$= \frac{\pi c\rho_0 v_0^2 k^6 a^8}{81}\int_{-1}^1 P_2^2(x)dx = \frac{\pi c\rho_0 v_0^2 k^6 a^8}{81}\left\|P_2\right\|^2 =$$

$$= \frac{\pi c\rho_0 v_0^2 k^6 a^8}{81}\frac{2}{2\cdot 2 + 1} = \frac{2\pi c\rho_0 v_0^2 k^6 a^8}{405}$$

$\overline{Y}$ - интенсивность излучения квадруполя,

Π - мощность излучения квадруполя.