Глава II. Некоторые классические задачи математической физики

§3. Задача о промерзании (задача о фазовом переходе, задача Стефана)

Построение решения задачи (1) – (4):

$$(2)\ \Rightarrow\ u_1 = A_1 + B_1\Phi\left(\frac{x}{2a_1\sqrt{t}}\right);\ u_2 = A_2 + B_2\Phi\left(\frac{x}{2a_2\sqrt{t}}\right)$$

$$x=0:\quad A_1 = T_1;\quad t=0:\quad A_2 + B_2 = T$$

$$(3)\ \Rightarrow\ A_1 + B_1\Phi\left(\frac{x}{2a_1\sqrt{t}}\right)_{x=\xi} = A_2 + B_2\Phi\left(\frac{x}{2a_2\sqrt{t}}\right)_{x=\xi} = 0$$

$$A_1 + B_1\Phi\left(\frac{x}{2a_1\sqrt{t}}\right)_{x=\xi} = 0\ \Rightarrow\ B_1 = -\frac{T_1}{\Phi\left(\frac{\xi}{2a_1\sqrt{t}}\right)}$$

$$A_2 = T - B_2 = -B_2\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)\ \Rightarrow$$

$$T = B_2\left(1-\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)\right)\ \Rightarrow\ B_2 = \frac{T}{1-\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)}$$

$$A_2 = T - B_2 = T - \frac{T}{1-\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)} = -\frac{T\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)}{1-\Phi\left(\frac{\xi}{2a_2\sqrt{t}}\right)}$$

$$(4)\ \Rightarrow\ \frac{\partial u_1}{\partial x} = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)}\quad \frac{\partial}{\partial x}\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a_1\sqrt{t}}} e^{-z^2}dz = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)}\frac{2}{\sqrt{\pi}}\frac{1}{2a_1\sqrt{t}}e^{-\frac{x^2}{4a_1^2 t}}$$

$$\frac{\partial u_2}{\partial x} = -\frac{T}{1-\Phi\left(\frac{\alpha}{2a_2}\right)}\frac{2}{\sqrt{\pi}}\frac{1}{2a_2\sqrt{t}}e^{-\frac{x^2}{4a_2^2 t}}$$

$$-k_1\frac{T_1}{a_1\sqrt{\pi t}\Phi\left(\frac{\alpha}{2a_1}\right)}e^{-\frac{\xi^2}{4a_1^2 t}} - k_2\frac{T}{a_2\sqrt{\pi t}\Phi\left(1-\Phi\left(\frac{\alpha}{2a_2}\right)\right)}e^{-\frac{\xi^2}{4a_2^2 t}} =$$

$$= \lambda\rho\frac{d}{dt}\left(\alpha\sqrt{t}\right) = \frac{\lambda\rho\alpha}{2\sqrt{t}}, \quad \text{где } \alpha = \frac{\xi}{\sqrt{t}} \quad \Rightarrow$$

$$\frac{k_1 T_1 e^{-\frac{\alpha^2}{4a_1^2}}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)} + \frac{k_2 T_1 e^{-\frac{\alpha^2}{4a_2^2}}}{a_2\left(1-\Phi\left(\frac{\alpha}{2a_2}\right)\right)} = -\lambda\rho\alpha\frac{\sqrt{\pi}}{2}$$

При $T = 0: \; B_2 = \frac{T}{1-\Phi\left(\frac{\alpha}{2a_2}\right)} = 0 \; \Rightarrow \; A_2 = T - B_2 = 0:$

$$A_1 = T_1; \; B_1 = -\frac{T_1}{\Phi\left(\frac{\alpha}{2a_1}\right)}; \; A_2 = 0; \; B_2 = 0$$

$$\frac{k_1 T_1 e^{-\frac{\alpha^2}{4a_1^2}}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)} = -\lambda\rho\alpha\frac{\sqrt{\pi}}{2} \qquad (10)$$

Положим $\beta = \frac{\alpha}{2a_1}, \; D = \frac{\lambda\rho a_1^2}{k_1 T_1}$

Тогда из (10) $\Rightarrow \frac{1}{\sqrt{\pi}}\frac{e^{-\beta^2}}{\Phi(\beta)} = -D\beta$